Top-K Ranking with a Monotone Adversary

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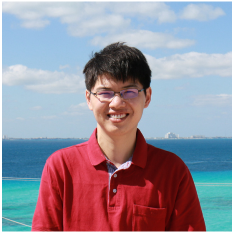

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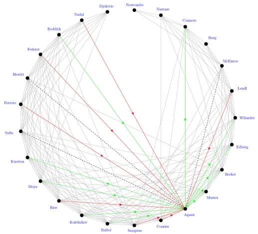
Ranking from pairwise comparisons



pairwise comparisons for ranking top tennis players

figure credit: Bozóki, Csató, Temesi

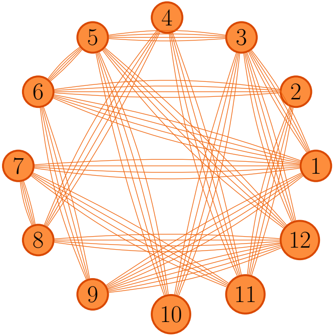
Bradley-Terry-Luce model: Assign latent score to each of n items $\boldsymbol{\theta}^\star = [\theta_1^\star, \cdots, \theta_n^\star]$ with

$$\mathbb{P}\left\{\text{item } j \text{ beats item } i\right\} = \frac{e^{\theta_i^\star}}{e^{\theta_i^\star} + e^{\theta_j^\star}}$$

Goal: identify the set of top-K items under minimal sample size

Sampling model

Sampling on comparison graph $\mathcal{G} = ([n], \mathcal{E})$: i, j are compared iff $(i, j) \in \mathcal{E}$



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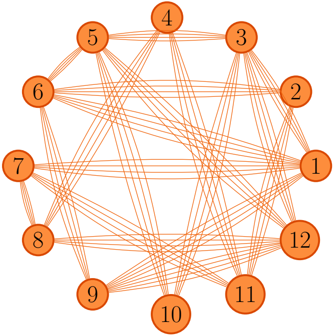

For each $(i, j) \in \mathcal{E}$, obtain L paired comparisons

$$y_{i,j}^{(l)} \overset{\text{ind.}}{=} \begin{cases} 1, & \text{with prob. } \frac{e^{\theta_j^\star}}{e^{\theta_i^\star} + e^{\theta_j^\star}} \\ 0, & \text{else} \end{cases} \qquad 1 \leq l \leq L$$

Prior art: MLE works for uniform sampling

- Uniform comparison graph: Erdős–Rényi graph $\mathcal{G}_{\mathrm{ER}} \sim \mathcal{G}(n, p)$

Theorem 1 (CFMW, AoS '19; CGZ, AoS '22)

When $p \gtrsim \frac{\log n}{n}$, regularized or unregularized MLE achieves optimal sample complexity for top-K ranking

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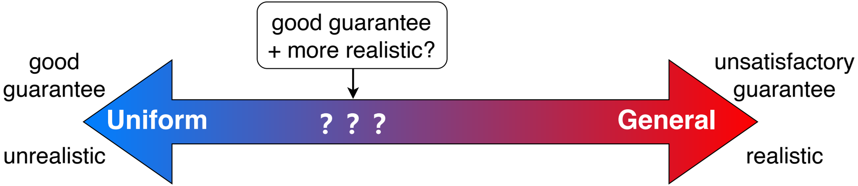
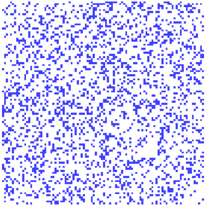


Top-K ranking with a monotone adversary

—aka semi-random adversary



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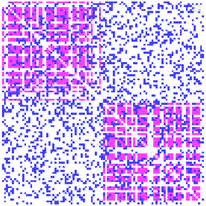

$\mathcal{G}_{\mathrm{SR}} = ([n], \mathcal{E}_{\mathrm{SR}})$ with added edges

Special case: non-uniform sampling $(i, j) \in \mathcal{E}$ with probability $p_{ij} \geq p$

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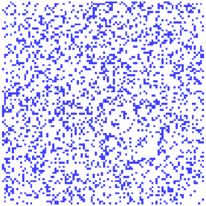

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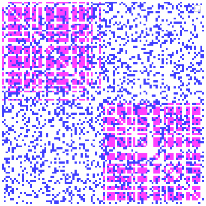

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Can we identify top-K items under monotone adversary?

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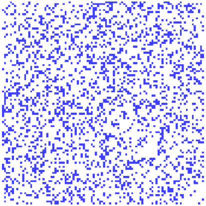

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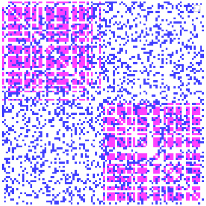

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Can we identify top-K items under monotone adversary? Not clear!

Intuition: mimicking oracle

If we have oracle knowledge of $\mathcal{E}_{\mathrm{ER}}$

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Can we find weights that mimic the above?

Optimal control of entrywise error

Theorem 2 (Yang, Chen, Oreccia, Ma, 2024)

When $p \gtrsim \frac{\log(n)}{n}$ *and* $npL \gtrsim \log^3(n)$*, with some proper reweighting, weighted MLE* $\widehat{\boldsymbol{\theta}}_w$ *obeys*

$$\|\widehat{\boldsymbol{\theta}}_w - \boldsymbol{\theta}^\star\|_\infty \lesssim \sqrt{\frac{\log(n)}{npL}}$$

will come back later to explain what is proper reweighting

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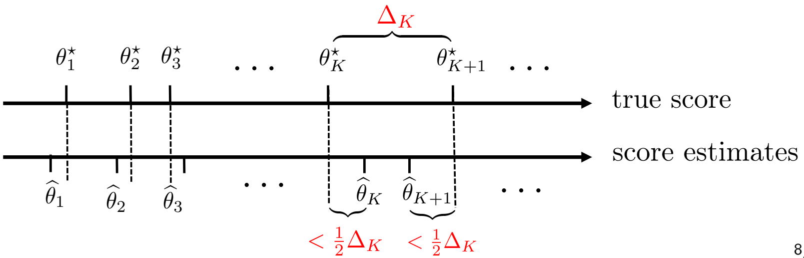
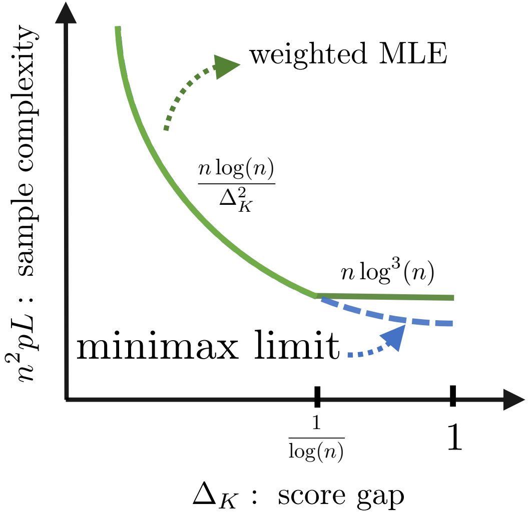


Near-optimal sample complexity



A few words about analysis

- ℓ_2 loss vs. ℓ_∞ loss
- Prior analysis e.g., leave-one-out analysis relies heavily on independence of edges, and also is not transparent in terms of graph properties

Master theorem for weighted MLE

- $w_{\max} := \max_{i,j} w_{ij}$ be the maximum weight
- $d_{\max} := \max_{i\in[n]} \sum_{j:j\neq i} w_{ij}$ be the maximum (weighted) degree
- Weighted graph Laplacian

$$\boldsymbol{L}_w := \sum_{(i,j):i>j} w_{ij}(\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top$$

Theorem 3 (Yang, Chen, Oreccia, Ma, 2024)

When graph is connected, as long as

$$L \gg \frac{w_{\max}(d_{\max})^4 \log^3(n)}{(\lambda_{n-1}(\boldsymbol{L}_w))^5},$$

with high probability, we have

$$\|\widehat{\boldsymbol{\theta}}_w - \boldsymbol{\theta}^\star\|_\infty \lesssim \sqrt{\frac{w_{\max}\log(n)}{\lambda_{n-1}(\boldsymbol{L}_w)L}}$$

Master theorem for weighted MLE

- LOO-free analysis that allows dependent edges
- Depends explicitly only on graph properties
- Applicable to other settings:
  "Random pairing MLE for estimation of item parameters in Rasch model", Yang and Ma, 2024

Optimization-based reweighting

- Master theorem motivates us to consider following SDP

$$\begin{aligned} \max_{\boldsymbol{w}} \quad & \lambda_{n-1}(\boldsymbol{L}_w) \\ \text{s.t.} \quad & \sum_i w_{ij} \le 2np \quad \text{for all } j \\ & 0 \le w_{ij} \le 1 \quad \text{for all } i, j \end{aligned}$$

- Since unit weights on $\mathcal{E}_{\mathrm{ER}}$ is feasible, we know the maximizer is at least as good as that for Erdős–Rényi graph

- Approximately solvable in near-linear time

Concluding remarks

Weighted MLE is statistically and computationally efficient for top-K ranking with monotone adversary

- Novel analysis of weighted MLE with general weights
- Efficient algorithm to approximately solve SDP-based reweighting

Future directions:

- Is weighted MLE necessary?
- Stronger adversary?

Papers:

- Y. Yang, A. Chen, L. Orecchia, C. Ma, "Top-K ranking with a monotone adversary," COLT, 2024
- Y. Yang, C. Ma, "Random pairing MLE for estimation of item parameters in Rasch model" arXiv, 2024